UDC 539.374

ON A CONSTRAINT IMPOSED BY THE CONDITION OF POSITIVE DISSIPATION ON THE BOUNDARY CONDITIONS FOR PLANE DEFORMATION OF A RIGID-PLASTIC BODY

PMM Vol. 41, № 1, 1977, pp. 188-192
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(Received December 11, 1975)

The effect of the requirement, that the dissipative function be positive on the boundary conditions for the velocities of a plastic body along its boundary with the rigid region, is investigated. A constraint is determined, which allows the elimination of solutions known to be incorrect without having to construct the velocity field. The incorrectness of the solution of the problem of a plane stamp acting on an edge of a convex piece, is shown.

Let an ideal rigid-plastic body undergo a plane deformation. We consider the line of discontinuity of velocities separating the rigid and the plastic region. We shall assume for convenience that the rigid body is stationary. As we know [1 — 5], the line of discontinuity of velocities represents the limiting position of a thin transitional layer across which the velocity varies in a continuous manner although rapidly. Using the condition that the dissipative function is positive

$$D = \sigma_\alpha \varepsilon_\alpha + \sigma_\beta \varepsilon_\beta + \tau_{\alpha\beta} \varepsilon_{\alpha\beta} \tag{1}$$

which for the thin layer in question assumes the form

$$\tau_{\alpha\beta} [v] > 0 \tag{2}$$

we also find that along the line of discontinuity of velocities we have

$$\varepsilon_{\alpha\beta} \sim [v], \quad \tau_{\alpha\beta} = \pm k, \quad k = \text{const} \tag{3}$$

and the sign of $\tau_{\alpha\beta}$ coincides with the sign of $[v]$, while $\sim$ is the proportionality sign [2].

Here $\sigma_\alpha, \ldots, \varepsilon_\alpha$ denote the components of the stress and rate of deformation tensors, $[v]$ is the velocity jump along the boundary of the stationary region and $[v] > 0$ provided that v increases together with the corresponding coordinate. For example, in the problem of impressing a stamp along the boundary $OBCDE$ (Fig. 1) $[v] > 0$. Consequently, by virtue of (2) and (3), we have $\varepsilon_{\alpha\beta} > 0$ and $\tau_{\alpha\beta} = k > 0$.

On the other hand, when the boundary value problems are solved for the velocities, the character of the rapid variation of the velocity within the thin layer, i.e. the sign of $\varepsilon_{\alpha\beta}$ at the boundary, is not taken into account. Therefore the velocity field does not need to satisfy the condition of positive dissipation at the arbitrarily close distances from the boundary.

Tne problem of convex stamp [3] is a well known example of this. Here the dissipative function is negative everywhere near the free surface (within the domain of the Cauchy problem) [6, 7] in spite of the fact that $[v] > 0$, and (2) holds along the boundary. This contradiction can be explained by the fact that the boundary value problem for the velocities can be formulated without taking into account the condition (2) along the boundary. Below we establish an additional restriction for the velocities along the boundary of the rigid region, and show that we can use the latter restriction to separate the solutions known in advance to be incorrect.

Let the boundary with the rigid region to be α-line where $\beta = 0$. We shall assume the rigid region to be stationary. Then we have the following relations for the velocity components along the slip lines, holding along the boundary:

$$v_\alpha = v_0 = \text{const}, \quad v_\beta = 0, \quad \beta = 0 \tag{4}$$

Consider in the plastic region an element bounded by the segments of the slip line ds_α and ds_β. We choose the directions of the α- and β-axes so that $\tau_{\alpha\beta} > 0$. Then, taking into account that $\sigma_\alpha = \sigma_\beta$, $\varepsilon_\alpha + \varepsilon_\beta = 0$, we can write (1) for the plastic deformation of the element as

$$D = \tau_{\alpha\beta}\varepsilon_{\alpha\beta} > 0 \tag{5}$$

Since we have here $\tau_{\alpha\beta} > 0$, (1) is equivalent to the inequality

$$\varepsilon_{\alpha\beta} = \frac{\partial v_\alpha}{\partial s_\beta} + \frac{\partial v_\beta}{\partial s_\alpha} + \frac{v_\alpha}{R_\alpha} + \frac{v_\beta}{R_\beta} > 0 \tag{6}$$

where $R_\alpha, \ldots$ are the radii of curvature of the slip lines, and $\partial / \partial s_\alpha, \ldots$ are the derivatives along the arcs of these lines. Formula (6) is the input formula for deriving the criterion of positiveness of the Green dissipation [8].

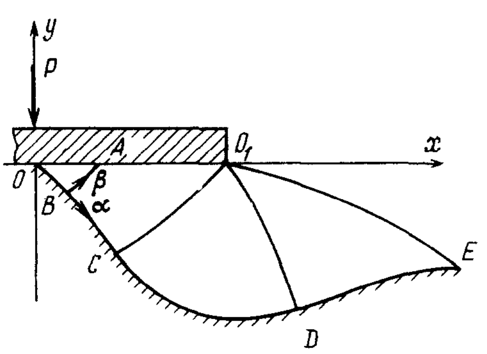

Fig. 1

Let us apply (6) to the element ds_α, ds_β lying infinitely near the boundary $\beta = 0$. Then, in accordance with (4) and (6), and taking into account the continuous character of the velocities and their derivatives, we obtain at $\beta = +0$

$$\varepsilon_{\alpha\beta} = \frac{\partial v_\alpha}{\partial s_\beta} + \frac{v_0}{R_\alpha} > 0 \tag{7}$$

(the case in which the medium moves as a solid body, is not considered).

When the grid of the slip lines is known, e.g. from the solution of stress equations, the first term in (7) will be the only unknown term. Let us express this term in terms of the boundary conditions. Consider the case when both families of the slip lines are curvilinear, and define the coordinates α and β for the slip lines as follows [4]:

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \frac{1}{2}\left[\left(\varphi \mp \frac{p}{2k}\right) \pm \frac{p_0}{2k}\right] \tag{8}$$

$$p = -\tfrac{1}{2}(\sigma_x + \sigma_y), \quad \varphi = \alpha + \beta \tag{9}$$

Here p_0 is the mean stress at the initial point O ($\alpha = 0$, $\beta = 0$) and φ is the angle between the tangents to the α-lines at the point in question, and at the point O. If the tangent lies in the anticlockwise direction from the tangent at the initial point, then $\varphi > 0$.

We note that $d\alpha = d\varphi$ and $d\beta = d\varphi$. Therefore the arc elements of the slip lines are connected with the radii of curvature by the following relations:

$$ds_\alpha = R_\alpha d\varphi = R_\alpha d\alpha, \quad ds_\beta = -R_\beta d\varphi = -R_\beta d_\beta \tag{10}$$

$$\frac{1}{R_\alpha} = \frac{\partial \varphi}{\partial s_\alpha}, \quad \frac{1}{R_\beta} = -\frac{\partial \varphi}{\partial s_\beta} \tag{11}$$

As before, we take the boundary with the rigid region ($\beta = 0$) along which the only coordinate that varies is α, and the slip line belonging to the family β, orthogonal to this boundary and passing through O, as the initial characterisitcs of α and β. The slip line is determined by the specific boundary conditions. During the motion along the boundary with the rigid region ($\beta = 0$) the coordinate α is, according to (9), equal to the angle of rotation of the tangent to the boundary counted from the initial point, and $\alpha > 0$ when the tangent rotates in the anticlockwise direction.

Using the present coordinate system, we have the following telegraph equation for the velocity component v_α:

$$\partial^2 v_\alpha / \partial\alpha\partial\beta + v_\alpha = 0$$

Integrating this equation with respect to α along the boundary, i.e. for $\beta = +0$, where according to (4) $v_\alpha = v_0$, we obtain

$$\left.\frac{\partial v_\alpha}{\partial\beta}\right|_{\beta=0} + v_0\alpha = \left(\frac{\partial v_\alpha}{\partial\beta}\right)^\circ, \quad v_0 = \text{const} \tag{12}$$

Here and henceforth the values of the quantities at $\alpha = 0$, $\beta = 0$ are denoted by the superscript $^\circ$.

Since in the coordinate system used the arc element of the slip line of family β is, as given by (11), $ds_\beta = -R_\beta d\beta$, consequently we find from (12), for $\beta = +0$, the following expression:

$$\frac{\partial v_\alpha}{\partial s_\beta} = -\frac{1}{R_\beta}\frac{\partial v_\alpha}{\partial\beta} = \frac{1}{R_\beta}\left[v_0\alpha - \left(\frac{\partial v_\alpha}{\partial\beta}\right)^\circ\right] = \frac{1}{R_\beta}\left[v_0\alpha + \left(R_\beta\frac{\partial v_\alpha}{\partial s_\beta}\right)^\circ\right] \tag{13}$$

Substituting (13) into (7), we obtain the restriction which imposes the requirement that the dissipation function (1) be positive on the boundary conditions along the boundary with the rigid stationary region

$$\frac{1}{R_\beta}\left[v_0\alpha + \left(R_\beta\frac{\partial v_\alpha}{\partial s_\beta}\right)^\circ\right] + \frac{v_0}{R_\alpha} > 1, \quad \beta = 0 \tag{14}$$

We note that R_β is a continuous function along the boundary, and the radius of curvature of the boundary may have a discontinuity.

We also note that if the family of the β-slip lines orthogonal to the boundary consists of straight lines, i.e. $1 / R_\beta = 0$, then (14) assumes the form

$$\left(\frac{\partial v_\alpha}{\partial s_\beta}\right)^\circ + \frac{v_0}{R_\alpha} > 0, \quad \beta = 0$$

The latter result can also be obtained directly from the Geiringer equations and the boundary conditions.

Example. We consider a problem of a smooth plane stamp acting upon an edge of a convex piece with symmetric curvilinear free contours [3]. Figure 1 shows the pattern of the slip lines under a two-sided pushing-out. We take the middle of the stamp base, i.e. the point O ($\alpha = 0$, $\beta = 0$) as the initial point.

The boundary conditions when the stamp is impressed with a unit velocity, will be

$$v_\alpha = \sqrt{2} = v_0, \quad v_\beta = 0, \quad \beta = 0 \tag{15}$$

along the boundary $OBCDE$ with the rigid region and

$$v_\alpha - v_\beta = \sqrt{2} = v_0 \tag{16}$$

under the stamp.

Let us find $\partial v_\alpha / \partial s_\beta$ for $\alpha = +0$, $\beta = +0$. To do this we consider an elementary right triangle OBA (Fig. 1) where $OB = ds_\alpha$ and $BA = ds_\beta$. We shall use the Geiringer equations [1]

$$dv_\alpha - v_\beta d\varphi = 0 \quad (\beta = \text{const})$$
$$dv_\beta + v_\alpha d\varphi = 0 \quad (\alpha = \text{const}) \tag{17}$$

At the point B ($\beta = 0$) we have, in accordance with (15), $v_\alpha = \sqrt{2} = v_0$. Therefore along $BA = ds_\beta$ we have, in accordance with the second equation of (17)

$$dv_\beta = -v_0 d\varphi \tag{18}$$

where $d\varphi$ is the angle of rotation of the tangent to the α-line along BA. Further, by virtue of (16) and (18), we have at the point A

$$v_\alpha = v_0 + v_\beta = v_0 - v_0 d\varphi$$

From here we obtain, taking into account that $v_\alpha = v_0$ at the point B, the following formula for dv_α along $BA = ds_\beta$, $dv_\alpha = v_0 - v_0 d\varphi - v_0 = -v_0 d\varphi$. Consequently, taking into account (11), we obtain

$$\left(\frac{\partial v_\alpha}{\partial s_\beta}\right)^\circ = -v_0 \frac{d\varphi}{ds_\beta} = \left(\frac{v_0}{R_\beta}\right)^\circ \tag{19}$$

Substitution of (19) into (14) yields the restriction which is imposed in this problem on the boundary conditions along the boundary with the rigid region, by the requirement that the dissipative function (1) be positive

$$\alpha + 1/R_\beta + 1/R_\alpha > 0, \quad \beta = 0 \tag{20}$$

As was shown in [3], in the present problem $R_\alpha < 0$ and $R_\beta < 0$ in the Cauchy domain (O_1DE in Fig. 1). Moreover, in the Cauchy domain the angle of rotation of the tangent to the line $\beta = 0$ (the boundary) is measured from the initial point O in the anticlockwise direction, i.e. $\alpha > 0$. Therefore the condition (20) does not hold here and the corresponding solution cannot be complete.

Note. We have said above that the formula (6) is the input formula for deriving the criterion of positiveness of the Green dissipation [8, 9]

$$U / R_\alpha - V / R_\beta > 0 \tag{21}$$

where U and V are radii of curvature of the mappings of the lines α and β onto the layout of velocities, and [9]

$$U = \frac{\partial v_\beta}{\partial \alpha} + v_\alpha, \quad V = \frac{\partial v_\alpha}{\partial \beta} - v_\beta \tag{22}$$

Although the Green criterion also utilizes the radii of curvature of the slip lines, the restriction (14) can only be obtained from (21) if we substitute the formulas (4) and (22) into (21). This is equivalent to returning to Eq. (6), which is the input equation for (21). All arguments following (7) remain valid.

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Translated by L. K.